Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 636–653
An International Journal computers & mathematics with applications
www.elsevier.com/locate/camwa

Existence of multiple positive solutions for a p-Laplacian system with sign-changing weight functions[☆]

Guoying Yang[a,b,*], Mingxin Wang[b]

[a] Department of Mathematics, Henan Polytechnic University, Jiaozuo 454000, PR China
[b] Department of Mathematics, Southeast University, Nanjing 210018, PR China

Received 15 June 2006; accepted 16 March 2007

Abstract

A p-Laplacian system with Dirichlet boundary conditions is investigated. By analysis of the relationship between the Nehari manifold and fibering maps, we will show how the Nehari manifold changes as λ, μ varies and try to establish the existence of multiple positive solutions.

Keywords: p-Laplacian system; Nehari manifold; Fibering maps; Multiple positive solutions

1. Introduction

By the fibering method, Drabek and Pohozaev in [1], Bozhkov and Mitidieri in [2] studied respectively the existence of multiple solutions to the following p-Laplacian single equation:

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u + c(x)|u|^{\alpha-1}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

and system:

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + (\alpha+1)c(x)|u|^{\alpha-1}u|v|^{\beta+1}, & x \in \Omega, \\ -\Delta_q v = \mu b(x)|v|^{q-2}v + (\beta+1)c(x)|u|^{\alpha+1}|v|^{\beta-1}v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $p, q > 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\Omega \subset \mathbb{R}^N$ is a bounded and connected domain with smooth boundary $\partial\Omega$, λ and μ are positive parameters, α and β are positive numbers. Functions $a(x), b(x), c(x) \in C(\bar{\Omega})$ are given functions which change sign on $\bar{\Omega}$.

☆ This work was supported by the National Natural Science Foundation of China 10471022, and the Ministry of Education of China Science and Technology Major Projects Grant 104090.

* Corresponding author at: Department of Mathematics, Henan Polytechnic University, Jiaozuo 454000, PR China.
E-mail address: ygyhfyg@yahoo.com.cn (G. Yang).

0898-1221/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2007.03.025

Recently, Brown and Zhang in [3] studied a special case $p = 2$ of the problem (1.1) by studying the Nehari manifold [4]. Exploiting the relationship between the Nehari manifold and fibering maps, they discussed how the Nehari manifold changes as λ changes and show how existence and non-existence results for positive solutions of this problem are linked to properties of the manifold.

Motivated by papers [1–3], in the present paper, we discuss the problem (1.2) again. The main purpose of this paper is show that how to use the similar idea and method of [3] to investigate the p-Laplacian system (1.2), and then get existence and non-existence results for positive solutions.

Let J be the Euler function associated with an elliptic problem on a Banach space X. If J is bounded below and J has a minimizer on X, then this minimizer is a critical point of J. So, it is a solution of the corresponding elliptic problem. However, the Euler function $J(u, v)$, associated with the problem (1.2), is not bounded below on the whole space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to solutions to (1.2). In this paper, we will show how the structure of the Nehari manifold is determined by the sign of $\int_\Omega c(x)\phi^{\alpha+1}\psi^{\beta+1}\mathrm{d}x$ and how the values of δ and σ are determined by the nature of the Nehari manifold. The functions ϕ and ψ will be given in hypothesis (H_3) in Section 2; δ and σ will be determined in Section 3.

For a single equation, the existence and multiplicity results have been obtained by using variational methods in [5–8], by the degree theory in [9] and by using global bifurcation theory in [10].

Systems involving quasilinear operators of p-Laplacian type have been studied by various authors [11,12]. Among other results, existence and non-existence theorems were obtained. For this purpose the method of sub–supersolutions, the blow-up method and the Mountain Pass Theorem have been used (see e.g. [11,13]).

The plan of this paper is as follows. In Section 2 we discuss the relation between the Nahari manifold and the fibering maps. In Section 3 we discuss the case $\lambda < \lambda_1(a), \mu < \mu_1(b)$ and show how the behavior of the manifold as $\lambda \nearrow \lambda_1(a), \mu \nearrow \mu_1(b)$ depends on the sign of $\int_\Omega c(x)\phi^{\alpha+1}\psi^{\beta+1}\mathrm{d}x$. In Section 4 we discuss the case $\lambda > \lambda_1(a), \mu > \mu_1(b)$ and obtain a new interpretation of δ, σ. In Section 5, we discuss the nature of the manifold for when (1.2) has no positive solutions.

2. Preliminaries

Let $\Omega \in \mathbb{R}^n$ be a bounded domain and $1 < p, q < \infty$. We define the Sobolev spaces $Y_p = W_0^{1,p}(\Omega)$ and $Y_q = W_0^{1,q}(\Omega)$ equipped with norms

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \qquad \|v\|_{1,q} = \left(\int_\Omega |\nabla u|^q \mathrm{d}x\right)^{\frac{1}{q}}.$$

Then we define $Y = Y_p \times Y_q$ and for $(u, v) \in Y$,

$$\|(u, v)\| = \|u\|_{1,p}^p + \|v\|_{1,q}^q.$$

Now consider the eigenvalue equation for the p-Laplacian operator:

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where $a(x) \in L^\infty(\Omega)$. We list the following lemma.

Lemma 2.1 (*[1,14]*).

(i) *There exists a number $\lambda_1 := \lambda_1(a) > 0$ such that*

$$\lambda_1 = \inf \frac{\int_\Omega |\nabla u|^p \mathrm{d}x}{\int_\Omega a(x)|u|^p \mathrm{d}x},$$

where the infimum is taken over $u \in Y_p$ such that $\int_\Omega a(x)|u|^p \mathrm{d}x > 0$.

(ii) *There exists a positive function $\phi \in Y_p \cap L^\infty(\Omega)$ which is a solution of* (2.1) *with $\lambda = \lambda_1$.*

(iii) *λ_1 is simple and isolated.*

Now we state the assumptions of this paper:

(H_1) $1 < p < \alpha + 1$, $1 < q < \beta + 1$. Define $d = (\alpha + 1)(\beta + 1) - (\alpha - p + 1)(\beta - q + 1) > 0$.
(H_2) $\frac{N-p}{p}(\alpha + 1) + \frac{N-q}{q}(\beta + 1) < N$, which implies that $\alpha + 1 < p^*$, $\beta + 1 < q^*$, where $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$ are the well-known critical exponents.
(H_3) The functions $a(x), b(x), c(x)$ are smooth functions which change sign in $\bar{\Omega}$.

Let $\lambda_1(a), \phi \in Y_p$ be the first eigenvalue and the corresponding eigenfunction of (2.1) respectively, and $\mu_1(b), \psi \in Y_q$ be the first eigenvalue and the first eigenfunction of

$$-\Delta_q v = \mu b(x)|v|^{q-2}v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega$$

respectively.

Definition 2.1 (*Weak Solution*). We say that $(u, v) \in Y$ is a weak solution of (1.2) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla z \mathrm{d}x = \lambda \int_\Omega a(x)|u|^{p-2}uz\mathrm{d}x + (\alpha + 1)\int_\Omega c(x)|u|^{\alpha-1}u|v|^{\beta+1}z\mathrm{d}x,$$
$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla w \mathrm{d}x = \mu \int_\Omega b(x)|v|^{q-2}uz\mathrm{d}x + (\beta + 1)\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1}vw\mathrm{d}x,$$

for any $(z, w) \in Y$.

It is clear that system (1.2) has a variational structure. Indeed, define

$$F(x, u, v) = \frac{\lambda}{p}a(x)|u|^p + \frac{\mu}{q}b(x)|v|^q + c(x)|u|^{\alpha+1}|v|^{\beta+1}$$

and let $J : Y \longrightarrow \mathbb{R}$ be defined by

$$J(u, v) = \frac{1}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + \frac{1}{q}\int_\Omega |\nabla v|^q \mathrm{d}x - \int_\Omega F(x, u, v)\mathrm{d}x,$$

or in a more detailed form,

$$J(u, v) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda a(x)|u|^p)\mathrm{d}x + \frac{1}{q}\int_\Omega (|\nabla v|^q - \mu b(x)|v|^q)\mathrm{d}x - \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.$$

Clearly, the critical points of J are the weak solutions of problem (1.2).

Then we introduce the following notation: for any functional $f : Y \longrightarrow \mathbb{R}$ we denote by $f'(u, v)(h_1, h_2)$ the Gateaux derivative of f at $(u, v) \in Y$ in the direction of $(h_1, h_2) \in Y$, and

$$f^{(1)}(u, v)h_1 = f'(u + \epsilon h_1, v)|_{\epsilon=0}, \qquad f^{(2)}(u, v)h_2 = f'(u, v + \delta h_2)|_{\delta=0}.$$

Let

$$S = \{(u, v) \in Y : J'(u, v)(u, v) = (J^{(1)}(u, v)u, J^{(2)}(u, v)v) = 0\}.$$

It is clear that all critical points of J must lie on S which is known as the Nehari manifold (see [4,15]). We will see below that local minimizers of J on S are usually critical points of J.

We simplify the notation by using

$$L(u) = \int_\Omega (|\nabla u|^p - \lambda a(x)|u|^p)\mathrm{d}x,$$
$$R(v) = \int_\Omega (|\nabla v|^q - \mu b(x)|v|^q)\mathrm{d}x,$$
$$G(u, v) = \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.$$

It is easy to see that $(u, v) \in S$ if and only if

$$\int_\Omega |\nabla u|^p \mathrm{d}x = \lambda \int_\Omega a(x)|u|^p \mathrm{d}x + (\alpha + 1)G(u, v), \tag{2.2}$$

$$\int_\Omega |\nabla v|^q \mathrm{d}x = \mu \int_\Omega b(x)|v|^q \mathrm{d}x + (\beta+1)G(u,v). \tag{2.3}$$

It is useful to understand S in terms of the stationary points of the form

$$I(t,s) = J(tu, sv), \quad t, s > 0.$$

We will refer to such maps as fibering maps. It is clear that if (u,v) is a local minimizer of J, then I has a local minimizer at $t = 1, s = 1$.

Theorem 2.1. *Let $(u,v) \in Y$, $u \not\equiv 0$, $v \not\equiv 0$ and $t, s > 0$. Then $(tu, sv) \in S$ if and only if $\frac{\partial I}{\partial t} = 0$, $\frac{\partial I}{\partial s} = 0$.*

Proof. The result is an immediate consequence of the fact that $\frac{\partial I}{\partial t} = J^{(1)}(tu,sv)u = \frac{1}{t}J^{(1)}(tu,sv)tu$, $\frac{\partial I}{\partial s} = J^{(2)}(tu,sv)v = \frac{1}{s}J^{(2)}(tu,sv)sv$. $\square$

Thus points in S correspond to stationary points of the map $I(t,s)$ and so it is natural to divide S into nine subsets. We have

$$\frac{\partial I}{\partial t} = t^{p-1}\int_\Omega |\nabla u|^p \mathrm{d}x - \lambda t^{p-1}\int_\Omega a(x)|u|^p \mathrm{d}x - (\alpha+1)t^\alpha s^{\beta+1}G(u,v), \tag{2.4}$$

$$\frac{\partial I}{\partial s} = s^{q-1}\int_\Omega |\nabla v|^q \mathrm{d}x - \mu s^{q-1}\int_\Omega b(x)|v|^q \mathrm{d}x - (\beta+1)t^{\alpha+1} s^{\beta}G(u,v). \tag{2.5}$$

Moreover,

$$\frac{\partial^2 I}{\partial t^2} = (p-1)t^{p-2}\int_\Omega |\nabla u|^p \mathrm{d}x - (p-1)\lambda t^{p-2}\int_\Omega a(x)|u|^p \mathrm{d}x - \alpha(\alpha+1)t^{\alpha-1} s^{\beta+1}G(u,v),$$

$$\frac{\partial^2 I}{\partial s^2} = (q-1)s^{q-2}\int_\Omega |\nabla v|^q \mathrm{d}x - (q-1)\mu s^{q-2}\int_\Omega b(x)|v|^q \mathrm{d}x - \beta(\beta+1)t^{\alpha+1} s^{\beta-1}G(u,v).$$

So,

$$\left.\frac{\partial^2 I}{\partial t^2}\right|_{(1,1)} = (p-1)\int_\Omega (|\nabla u|^p - \lambda a(x)|u|^p)\mathrm{d}x - \alpha(\alpha+1)G(u,v),$$

$$\left.\frac{\partial^2 I}{\partial s^2}\right|_{(1,1)} = (q-1)\int_\Omega (|\nabla v|^q - \mu b(x)|v|^q)\mathrm{d}x - \beta(\beta+1)G(u,v).$$

Hence, we define

$$S_+^+ = \left\{(u,v) \in S : \left.\frac{\partial^2 I}{\partial t^2}\right|_{(1,1)} > 0,\ \left.\frac{\partial^2 I}{\partial s^2}\right|_{(1,1)} > 0\right\},$$

$$S_-^- = \left\{(u,v) \in S : \left.\frac{\partial^2 I}{\partial t^2}\right|_{(1,1)} < 0,\ \left.\frac{\partial^2 I}{\partial s^2}\right|_{(1,1)} < 0\right\},$$

$$S_0^0 = \left\{(u,v) \in S : \left.\frac{\partial^2 I}{\partial t^2}\right|_{(1,1)} = 0,\ \left.\frac{\partial^2 I}{\partial s^2}\right|_{(1,1)} = 0\right\},$$

$$S_-^+ = \left\{(u,v) \in S : \left.\frac{\partial^2 I}{\partial t^2}\right|_{(1,1)} > 0,\ \left.\frac{\partial^2 I}{\partial s^2}\right|_{(1,1)} < 0\right\}.$$

Similarly, we can define S_0^+, S_+^0, S_0^-, S_-^0 and S_+^-. Since $(u,v) \in S$, (2.2) and (2.3) hold, which implies

$$S_+^+ = \{(u,v) \in S : (\alpha+1)(p-1-\alpha)G(u,v) > 0, (\beta+1)(q-1-\beta)G(u,v) > 0\}.$$

Since $p-1 < \alpha$, $q-1 < \beta$,

$$S_+^+ = \{(u,v) \in S : G(u,v) < 0\}.$$

Similarly,

$$S_-^- = \{(u, v) \in S : G(u, v) > 0\},$$
$$S_0^0 = \{(u, v) \in S : G(u, v) = 0\}.$$

Moreover, $S_-^+, S_+^-, S_0^+, S_+^0, S_0^-, S_-^0$ are empty since $p - 1 < \alpha$, $q - 1 < \beta$. So, S is divided into three subsets S_+^+, S_-^- and S_0^0. We denote these simply as S_+, S_-, S_0 respectively. Then we have

Theorem 2.2. *Let $(u, v) \in S$. Then*

(i) $\frac{\partial I}{\partial t}|_{(1,1)} = 0$, $\frac{\partial I}{\partial s}|_{(1,1)} = 0$.

(ii) $(u, v) \in S_+, S_-, S_0$ *if and only if* $\frac{\partial^2 I}{\partial t^2}|_{(1,1)} > 0$, $\frac{\partial^2 I}{\partial s^2}|_{(1,1)} > 0$; $\frac{\partial^2 I}{\partial t^2}|_{(1,1)} < 0$, $\frac{\partial^2 I}{\partial s^2}|_{(1,1)} < 0$; $\frac{\partial^2 I}{\partial t^2}|_{(1,1)} = 0$, $\frac{\partial^2 I}{\partial s^2}|_{(1,1)} = 0$ *respectively.*

The following theorem shows that minimizers on S are usually critical points for J.

Theorem 2.3. *Suppose that (u_0, v_0) is a local minimizer for J on S and $(u_0, v_0) \notin S_0$. Then $J'(u_0, v_0) = 0$.*

Proof. If $(u_0, v_0) \in Y$ is a local minimizer for J on S, then (u_0, v_0) is a solution of the optimization problem:

Find the minimizer of $J(u, v)$ subject to $E_1(u, v) = 0$, $E_2(u, v) = 0$, where

$$E_1(u, v) = \int_\Omega (|\nabla u|^p - \lambda a(x)|u|^p)\mathrm{d}x - (\alpha + 1)G(u, v),$$
$$E_2(u, v) = \int_\Omega (|\nabla v|^q - \mu b(x)|v|^q)\mathrm{d}x - (\beta + 1)G(u, v).$$

Hence, by the theory of Lagrange multipliers, there exists $m_1, m_2 \in \mathbb{R}$ such that

$$J'(u_0, v_0) = m_1 E_1'(u_0, v_0) + m_2 E_2'(u_0, v_0),$$

and thus

$$J'(u_0, v_0)(u_0, v_0) = m_1 E_1'(u_0, v_0)(u_0, v_0) + m_2 E_2'(u_0, v_0)(u_0, v_0).$$

Since $(u_0, v_0) \in S$, $J'(u_0, v_0)(u_0, v_0) = 0$, then

$$\begin{cases} m_1 \left(p \int_\Omega |\nabla u_0|^p \mathrm{d}x - p\lambda \int_\Omega a(x)|u_0|^p \mathrm{d}x - (\alpha + 1)^2 G(u_0, v_0) \right) + m_2(\alpha + 1)(\beta + 1)G(u_0, v_0) = 0, \\ m_2 \left(q \int_\Omega |\nabla v_0|^q \mathrm{d}x - q\mu \int_\Omega b(x)|v_0|^q \mathrm{d}x - (\beta + 1)^2 G(u_0, v_0) \right) + m_1(\alpha + 1)(\beta + 1)G(u_0, v_0) = 0. \end{cases} \tag{2.6}$$

From (2.2) and (2.3), we know that (2.6) is equivalent to

$$\begin{cases} (m_1 p - m_1(\alpha + 1) + m_2(\beta + 1))G(u_0, v_0) = 0, \\ (m_2 q - m_2(\beta + 1) + m_1(\alpha + 1))G(u_0, v_0) = 0. \end{cases}$$

Thus, if $(u_0, v_0) \notin S_0$, then $G(u_0, v_0) \neq 0$. So

$$\begin{cases} m_1 p - m_1(\alpha + 1) + m_2(\beta + 1) = 0, \\ m_2 q - m_2(\beta + 1) + m_1(\alpha + 1) = 0, \end{cases}$$

which implies $m_1 = m_2 = 0$. The proof is complete. □

It is easy to see that (2.4) and (2.5) are equivalent to

$$\begin{cases} \dfrac{\partial I}{\partial t} = t^{p-1} L(u) - (\alpha + 1)t^\alpha s^{\beta+1} G(u, v), \\ \dfrac{\partial I}{\partial s} = s^{q-1} R(v) - (\beta + 1)t^{\alpha+1} s^\beta G(u, v). \end{cases}$$

If $\frac{\partial I}{\partial t} = \frac{\partial I}{\partial s} = 0$, then

$$\begin{cases} t^{p-1-\alpha} = (\alpha+1)s^{\beta+1}\dfrac{G(u,v)}{L(u)}, \\ s^{q-1-\beta} = (\beta+1)t^{\alpha+1}\dfrac{G(u,v)}{R(v)}. \end{cases}$$

Thus, if $L(u)$, $R(v)$ and $G(u,v)$ have the same signs, then $I(t,s)$ has exactly one turning point at

$$t = \left(\frac{(\alpha+1)^{\beta-q+1}|R(v)|^{\beta+1}}{(\beta+1)^{\beta+1}|G(u,v)|^{q}|L(u)|^{\beta-q+1}}\right)^{1/d}, \qquad s = \left(\frac{(\beta+1)^{\alpha-p+1}|L(u)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}|G(u,v)|^{p}|R(v)|^{\alpha-p+1}}\right)^{1/d}, \tag{2.7}$$

where $d = (\alpha+1)(\beta+1) - (\alpha+1-p)(\beta+1-q) > 0$. By calculation, t, s have the following property:

$$t(k_1 u, k_2 v) = \frac{1}{k_1} t(u,v), \quad (k_1 u, k_2 v) = \frac{1}{k_2} s(u,v),\ k_1, k_2 > 0.$$

So

$$(t(k_1 u, k_2 v)k_1 u, s(k_1 u, k_2 v)k_2 v) = (t(u,v)u, s(u,v)v),$$

which is important in Sections 3 and 4.

If $L(u)$, $R(v)$ and $G(u,v)$ have opposite signs, then $I(t,s)$ has no turning points. To get our results, we just verify that $L(u)$, $R(v)$ and $G(u,v)$ have the same signs.

We define

$$\Lambda^+ = \left\{(u,v) \in Y : \|u\|_{1,p} = \|v\|_{1,q} = 1, L(u) > 0, R(v) > 0\right\}.$$

Similarly, by replacing "$>$" by "$<$" ("$=$"), we define $\Lambda^-(\Lambda^0)$.

Define

$$B^+ = \{(u,v) \in Y, \|u\|_{1,p} = \|v\|_{1,q} = 1, G(u,v) > 0\}.$$

Similarly, by replacing "$>$" by "$<$" ("$=$"), we define $B^-(B^0)$.

Thus, if $(u,v) \in \Lambda^+ \cap B^+$, $I(t,s) > 0$ for t, s small and positive but $I(t,s) \to -\infty$ as $t \to \infty$ and $s \to \infty$; also $I(t,s)$ has a unique (maximum) stationary point at $(t(u,v), s(u,v))$ and $(t(u,v)u, s(u,v)v) \in S_-$. Similarly, if $(u,v) \in \Lambda^- \cap B^-$, $I(t,s) < 0$ for t, s small and positive but $I(t,s) \to \infty$ as $t \to \infty$ and $s \to \infty$; also $I(t,s)$ has a unique (minimum) stationary point at $(t(u,v), s(u,v))$ and $(t(u,v)u, s(u,v)v) \in S_+$.

Thus, if $(u,v) \in Y$ and $u \not\equiv 0$, $v \not\equiv 0$, then:

(i) a multiple of u and a multiple of v lie in S_- if and only if $(\frac{u}{\|u\|_{1,p}}, \frac{v}{\|v\|_{1,q}}) \in \Lambda^+ \cap B^+$;
(ii) a multiple of u and a multiple of v lie in S_+ if and only if $(\frac{u}{\|u\|_{1,p}}, \frac{v}{\|v\|_{1,q}}) \in \Lambda^- \cap B^-$;
(iii) when (u,v) is neither in $\Lambda^+ \cap B^+$ nor in $\Lambda^- \cap B^-$, no multiple (u,v) lies in S.

3. The case when $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$

Suppose that $0 < \lambda < \lambda_1(a)$, $0 < \mu < \mu_1(b)$. It is easy to deduce by contradiction with the first eigenvalue that there exists $\delta_0, \delta_1 > 0$ such that

$$L(u) \ge \delta_0 \|u\|_{1,p}^p, \qquad R(v) \ge \delta_1 \|v\|_{1,q}^q, \quad \forall\, (u,v) \in Y.$$

Thus Λ^- and Λ^0 are empty and so S_+ is empty and $S_0 = \{u = v = 0\}$. Moreover,

$$S_- = \{(t(u,v)u, s(u,v)v), (u,v) \in B^+\}, \qquad S = S_- \cup S_0.$$

Theorem 3.1. *Assume that $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$. Then (1.2) has at least one positive solution.*

Proof. We investigate the behavior of J on S_-. Clearly $J(u,v) \geq 0$ if $(u,v) \in S_-$ and so $J(u,v)$ is bounded below by 0 on S_-. We now show that $\inf_{(u,v)\in S_-} J(u,v) > 0$. Suppose $(u,v) \in S_-$. Let $\bar{u} = \frac{u}{\|u\|_{1,p}}$, $\bar{v} = \frac{v}{\|v\|_{1,q}}$, then $(\bar{u},\bar{v}) \in \Lambda^+ \cap B^+$ and $u = t(\bar{u},\bar{v})\bar{u}$, $v = s(\bar{u},\bar{v})\bar{v}$, where t and s are determined by (2.7).

Now, for $C_1 > 0$,

$$G(\bar{u},\bar{v}) = \int_\Omega c(x)|\bar{u}|^{\alpha+1}|\bar{v}|^{\beta+1}\mathrm{d}x \leq C_1\|\bar{u}\|_{1,p}^{\alpha+1}\|\bar{v}\|_{1,q}^{\beta+1}. \tag{3.1}$$

Indeed, by condition (H_2) we have

$$\frac{Np}{(\alpha+1)(N-p)} - \frac{Nq}{Nq-(\beta+1)(N-q)} > 0.$$

So, there exists ϵ_0 such that

$$0 < \epsilon_0 < \frac{Np}{(\alpha+1)(N-p)} - \frac{Nq}{Nq-(\beta+1)(N-q)},$$

which implies

$$\frac{(\beta+1)(p^* - \epsilon_0(\alpha+1))}{p^* - (\epsilon_0+1)(\alpha+1)} < q^* = \frac{Nq}{N-q}.$$

Then, using the Hölder inequality and the Sobolev inequality, we get

$$\begin{aligned} G(\bar{u},\bar{v}) &\leq C\left(\int_\Omega [(|\bar{u}|)^{\alpha+1}]^{\frac{p^*}{\alpha+1}-\epsilon_0}\right)^{\frac{\alpha+1}{p^*-\epsilon_0(\alpha+1)}}\left(\int_\Omega [(|\bar{v}|)^{\beta+1}]^{\frac{p^*-\epsilon_0(\alpha+1)}{p^*-(\epsilon_0+1)(\alpha+1)}}\right)^{\frac{p^*-(\epsilon_0+1)(\alpha+1)}{p^*-\epsilon_0(\alpha+1)}} \\ &\leq \left(\int_\Omega |\nabla\bar{u}|^p\mathrm{d}x\right)^{\frac{\alpha+1}{p}}\left(\int_\Omega |\nabla\bar{v}|^q\mathrm{d}x\right)^{\frac{\beta+1}{q}} \\ &= C_1\|\bar{u}\|_{1,p}^{\alpha+1}\|\bar{v}\|_{1,q}^{\beta+1} = C_1. \end{aligned}$$

Hence,

$$J(u,v) = J(t(\bar{u},\bar{v})\bar{u}, s(\bar{u},\bar{v})\bar{v}) = K\frac{(L(\bar{u}))^{(\alpha+1)q/d}(R(\bar{v}))^{(\beta+1)p/d}}{(G(\bar{u},\bar{v}))^{pq/d}},$$

where

$$K = \left(\frac{(\alpha+1)^{\frac{(\beta+1-q)p}{d}}}{p(\beta+1)^{\frac{p(\beta+1)}{d}}} + \frac{(\beta+1)^{\frac{(\alpha+1-p)q}{d}}}{q(\alpha+1)^{\frac{q(\alpha+1)}{d}}} - \frac{1}{(\alpha+1)^{\frac{q(\alpha+1)}{d}}(\beta+1)^{\frac{p(\beta+1)}{d}}}\right) \times \operatorname{sign}(G(\bar{u},\bar{v})).$$

Since $(\bar{u},\bar{v}) \in \Lambda^+ \cap B^+$, we have $K > 0$ and so

$$J(u,v) \geq K\frac{(\delta_0)^{\frac{(\alpha+1)q}{d}}\delta_1^{\frac{(\beta+1)p}{d}}}{C_1^{\frac{pq}{d}}}.$$

Hence, $\inf_{(u,v)\in S_-} J(u,v) > 0$.

We now show that there exists a minimizer on S_- which is a critical point of $J(u,v)$ and so a non-trivial solution of (1.2). Let $\{(u_n,v_n)\} \in S_-$ be a minimizer sequence, i.e., $\lim_{n\to\infty} J(u_n,v_n) = \inf_{(u,v)\in S_-} J(u,v)$. Since

$$\begin{aligned} J(u_n,v_n) &= \frac{1}{p}L(u_n) + \frac{1}{q}R(v_n) - G(u_n,v_n) \\ &= \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right)G(u_n,v_n) \\ &= \left(\frac{1}{p} + \frac{\beta+1}{q(\alpha+1)} - \frac{1}{\alpha+1}\right)L(u_n) \\ &\geq \left(\frac{1}{p} + \frac{\beta+1}{q(\alpha+1)} - \frac{1}{\alpha+1}\right)\delta_0\|u_n\|_{1,p}^p, \end{aligned}$$

and, similarly,

$$\begin{aligned} J(u_n, v_n) &= \left(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\right) R(v_n) \\ &\geq \left(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\right) \delta_1 \|v_n\|_{1,q}^q, \end{aligned}$$

then (u_n, v_n) is bounded in Y; we can pass to a subsequence if necessary and have that

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{in } Y, \\ u_n \to u_0 \quad \text{in } L^p(\Omega) \cap \times L^{\alpha+1}(\Omega), \qquad v_n \to v_0 \quad \text{in } L^q(\Omega) \cap \times L^{\beta+1}(\Omega), \\ \text{also,} \quad G(u_n, v_n) \to G(u_0, v_0). \end{cases} \tag{3.2}$$

Now,

$$\begin{aligned} 0 < \lim_{n\to\infty} J(u_n, v_n) &= \lim_{n\to\infty} \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) G(u_n, v_n) \\ &= \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) G(u_0, v_0) \end{aligned}$$

and so $u_0 \neq 0$, $v_0 \neq 0$. Since $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$, we have

$$L(u_0) > 0, \qquad R(v_0) > 0.$$

Hence, a multiple of u_0 and a multiple of v_0 lie in $\Lambda^+ \cap B^+$. From the lower semi-continuity, $\|u_0\|_{1,p} \leq \lim_{n\to\infty} \|u_n\|_{1,p}$, $\|v_0\|_{1,q} \leq \lim_{n\to\infty} \|v_n\|_{1,q}$. We now show that

$$\|u_0\|_{1,p} = \lim_{n\to\infty} \|u_n\|_{1,p}, \qquad \|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}. \tag{3.3}$$

If this is not true, then three cases occur:

(a) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$, $\|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}$;
(b) $\|u_0\|_{1,p} = \lim_{n\to\infty} \|u_n\|_{1,p}$, $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$;
(c) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$, $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$.

If (a) occurs, then

$$L(u_0) - (\alpha+1)G(u_0, v_0) < \lim_{n\to\infty} (L(u_n) - (\alpha+1)G(u_n, v_n)) = 0.$$

We will obtain a contradiction by considering the fibering map $I(t, s)$. We have

$$\left.\frac{\partial I}{\partial t}\right|_{(1,1)} = L(u_0) - (\alpha+1)G(u_0, v_0) < 0.$$

From the analysis of Section 2, there exists $(x_0, y_0) \neq (1, 1)$ such that $\frac{\partial I}{\partial t}|_{(x_0,y_0)} = 0$, $\frac{\partial I}{\partial s}|_{(x_0,y_0)} = 0$, i.e. $(x_0u_0, y_0v_0) \in S_-$. Now, $(x_0u_n, y_0v_n) \rightharpoonup (x_0u_0, y_0v_0)$ in Y. Moreover, as $(u_n, v_n) \in S_-$, the map $I(t, s)$ attains its maximum at $t = s = 1$. Hence,

$$J(x_0u_0, y_0v_0) < \lim_{n\to\infty} J(x_0u_n, y_0v_n) \leq \lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in S_-} J(u, v)$$

and this is a contradiction.

Similarly, if (b) or (c) occurs, we also get the contradiction.

It follows easily from (3.3) that

$$L(u_0) - (\alpha+1)G(u_0, v_0) = 0, \qquad R(v_0) - (\beta+1)G(u_0, v_0) = 0.$$

and so $(u_0, v_0) \in S_-$. Also $J(u_0, v_0) = \lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in S_-} J(u, v)$ and so (u_0, v_0) is a minimizer on S_-. Since $G(u_0, v_0) > 0$, $(u_0, v_0) \notin S_0$ and by Theorem 2.3 (u_0, v_0) is a critical point of $J(u, v)$. Since $J(|u|, |v|) = J(u, v)$, we may assume that (u_0, v_0) is positive. $\square$

We now consider what happens as $\lambda \nearrow \lambda_1(a), \mu \nearrow \mu_1(b)$. The sign of

$$G(\phi, \psi) = \int_\Omega c(x)\phi^{\alpha+1}\psi^{\beta+1}\mathrm{d}x > 0$$

will play an important role.

Theorem 3.2. *Suppose that* $\int_\Omega c(x)\phi^{\alpha+1}\psi^{\beta+1}\mathrm{d}x > 0$. *Then*

$$\lim_{\lambda \nearrow \lambda_1(a)} \inf_{(u,v)\in S_-} J(u, v) = 0 \quad \textit{and} \quad \lim_{\mu \nearrow \mu_1(b)} \inf_{(u,v)\in S_-} J(u, v) = 0.$$

Proof. Without loss of generality, we assume that $\|\phi\|_{1,p} = \|\psi\|_{1,q} = 1$. It is clear that $(\phi, \psi) \in B^+$. Since $\lambda < \lambda_1(a)$, $\mu < \mu_1(b)$, we get $(\phi, \psi) \in \Lambda^+$ and so $(\phi, \psi) \in \Lambda^+ \cap B^+$. Hence, $(t(\phi, \psi)\phi, s(\phi, \psi)\psi) \in S_-$ and

$$\begin{aligned} J(t(\phi, \psi)\phi, s(\phi, \psi)\psi) = \quad & K\frac{\left(\int_\Omega(\lambda_1 - \lambda)a(x)\phi^p\mathrm{d}x\right)^{\frac{(\alpha+1)q}{d}}\left(\int_\Omega(\mu_1 - \mu)b(x)\psi^q\mathrm{d}x\right)^{\frac{(\beta+1)p}{d}}}{(G(\phi, \psi))^{pq/d}} \\ = \quad & K(\lambda_1 - \lambda)^{\frac{(\alpha+1)q}{d}}(\mu_1 - \mu)^{\frac{(\beta+1)p}{d}}\frac{\left(\int_\Omega a(x)\phi^p\mathrm{d}x\right)^{\frac{(\alpha+1)q}{d}}\left(\int_\Omega b(x)\psi^q\mathrm{d}x\right)^{\frac{(\beta+1)p}{d}}}{(G(\phi, \psi))^{pq/d}} \\ \longrightarrow \; & 0 \end{aligned}$$

as $\lambda \nearrow \lambda_1(a)$ or $\mu \nearrow \mu_1(b)$.

Since $0 < \inf_{(u,v)\in S_-} J(u, v) \le J(t(\phi, \psi)\phi, s(\phi, \psi)\psi)$, it follows that the conclusion is true. □

4. The case when $\lambda > \lambda_1(a)$, $\mu > \mu_1(b)$

If $\lambda > \lambda_1(a)$, $\mu > \mu_1(b)$, then we have

$$L(\phi) = \int_\Omega (|\nabla\phi|^p - \lambda a(x)\phi^p)\mathrm{d}x = (\lambda_1(a) - \lambda)\int_\Omega a(x)\phi^p\mathrm{d}x < 0,$$
$$R(\psi) = \int_\Omega (|\nabla\psi|^q - \lambda b(x)\psi^q)\mathrm{d}x = (\mu_1(b) - \mu)\int_\Omega b(x)\psi^q\mathrm{d}x < 0.$$

So $(\phi, \psi) \in \Lambda^-$. Hence if $G(\phi, \psi) < 0$, then $(\phi, \psi) \in \Lambda^- \cap B^-$ and so S_+ is non-empty. Thus, S may consists of two distinct components in this case which makes it possible to prove the existence of at least two positive solutions by showing that $J(u, v)$ has a minimizer on each component.

In the following lemma and theorems, we show that $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$ is an important condition for establishing the existence of minimizers.

Lemma 4.1. *Suppose that* $G(\phi, \psi) < 0$. *Then there exists* $\delta > 0, \sigma > 0$ *such that* $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$ *whenever* $\lambda_1(a) \le \lambda < \lambda_1(a) + \delta$, $\mu_1(b) \le \mu < \mu_1(b) + \sigma$.

Proof. Suppose that the result is false. Then there exist sequences $\{(\lambda_n, \mu_n)\}$ and $\{(u_n, v_n)\}$ such that $\|u_n\|_{1,p} = 1$, $\|v_n\|_{1,q} = 1$, $\lambda_n \searrow \lambda_1(a)$, $\mu_n \searrow \mu_1(b)$ and

$$L(u_n) = \int_\Omega (|\nabla u_n|^p - \lambda_n a(x)|u_n|^p)\mathrm{d}x \le 0, \tag{4.1}$$

$$R(v_n) = \int_\Omega (|\nabla v_n|^q - \mu_n b(x)|v_n|^q)\mathrm{d}x \le 0, \tag{4.2}$$

$$G(u_n, v_n) = \int_\Omega c(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x \ge 0. \tag{4.3}$$

Since $\{(u_n, v_n)\}$ is bounded, we may assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in Y and then $G(u_n, v_n) \to G(u_0, v_0)$. From the lower semi-continuity of norms in Y, we assert that (3.3) holds. If this is not true, we may assume that

$$\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}, \qquad \|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}.$$

Then by (4.1)

$$0 \le L(u_0) < \lim_{n\to\infty} L(u_n) < 0,$$

which is impossible. Hence, (3.3) holds and so $\|u_0\|_{1,p} = \|v_0\|_{1,q} = 1$. From (4.1)–(4.3), it follows that

$$L(u_0) \le 0, \qquad R(v_0) \le 0, \qquad G(u_0, v_0) \ge 0.$$

The first two inequalities imply that $u_0 = k_1\phi$, $v_0 = k_2\psi$. But from the last inequality and the condition $G(\phi, \psi) < 0$, we deduce that $k_1 = 0$ or $k_2 = 0$ or $k_1 = k_2 = 0$. It is impossible for $\|u_0\|_{1,p} = \|v_0\|_{1,q} = 1$. □

Lemma 4.2. *Suppose that $G(\phi, \psi) < 0$ and $(u, v) \in S_-$. Then there exists $\delta_1, \delta_2 > 0$ and $\delta, \sigma > 0$ such that*

$$L(u) \ge \delta_1 \|u\|_{1,p}^p, \qquad R(v) \ge \delta_2 \|v\|_{1,q}^q$$

whenever $\lambda_1(a) \le \lambda < \lambda_1(a) + \delta$, $\mu_1(b) \le \mu < \mu_1(b) + \sigma$.

Proof. The proof of Lemma 4.2 is similar to the proof of Theorem 1(iii) in [7]. □

We next show that if $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$, it is possible to obtain more information about the nature of the Nehari manifold.

Theorem 4.1. *Assume that $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$. Then*

(i) $S_0 = \{(u, v) \in Y : u = 0,\ \text{or } v = 0,\ \text{or } u = v = 0\}$
(ii) *For any $(u, v) \in S_0$, we have $(u, v) \notin \overline{S_-}$ and S_- is closed.*
(iii) *S_- and S_+ are separated, i.e. $\overline{S_-} \cap \overline{S_+} = \emptyset$.*

Proof. (i) Suppose that $(u_0, v_0) \in S_0$ but $u_0 \ne 0$, $v_0 \ne 0$. Then

$$\left(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}\right) \in \Lambda^0 \cap B^0 \subset \overline{\Lambda^-} \cap \overline{B^+} = \emptyset,$$

which is impossible.

(ii) If the conclusion is not true, then $(u, v) \in \overline{S_-}$. We divide the proof into three cases:

(a) $u = 0, v \ne 0$; (b) $u \ne 0, v = 0$; (c) $u = 0, v = 0$.

If the case (a) occurs, there exists $\{(u_n, v_n)\} \in S_-$ such that $\lim_{n\to\infty}(u_n, v_n) = (0, v)$ in Y. Hence, as $n \to \infty$,

$$0 < L(u_n) = (\beta + 1)G(u_n, v_n) \longrightarrow 0,$$
$$0 < R(v_n) = (\alpha + 1)G(u_n, v_n) \longrightarrow 0.$$

Let $\bar{u}_n = \frac{u_n}{\|u_n\|_{1,p}}$. Then we may assume that $\bar{u}_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ and $\bar{u}_n \to u_0$ in $L^p(\Omega)$ and in $L^{\alpha+1}(\Omega)$.

Clearly, as $n \to \infty$,

$$0 < L(\bar{u}_n) = (\alpha + 1)\|u_n\|_{1,p}^{\alpha+1-p} \int_\Omega c(x)|\bar{u}_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x \longrightarrow 0. \tag{4.4}$$

Since $\|\bar{u}_n\|_{1,p} = 1$, we deduce that

$$0 = \lim_{n\to\infty} L(\bar{u}_n) = 1 - \lambda \lim_{n\to\infty} \int_\Omega a(x)|\bar{u}_n|^p\mathrm{d}x = 1 - \lambda \int_\Omega a(x)|u_0|^p\mathrm{d}x,$$

and so $u_0 \ne 0$. Moreover,

$$L(u_0) \le \lim_{n\to\infty} L(u_n) = 0,$$

which implies

$$\int_\Omega \left(\left|\nabla \frac{u_0}{\|u_0\|_{1,p}}\right|^p - \lambda a(x)\left|\frac{u_0}{\|u_0\|_{1,p}}\right|^p\right)\mathrm{d}x \le 0.$$

Since $(0, v) \in S_0$, it follows that $R(v) = 0$. Hence $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v}{\|v\|_{1,q}}) \in \overline{\Lambda^-}$. From (4.4), it is easy to obtain that $\int_\Omega c(x)|u_0|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x \geq 0$. Consequently $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v}{\|v\|_{1,q}}) \in \overline{B^+}$. Therefore,

$$\left(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v}{\|v\|_{1,q}}\right) \in \overline{\Lambda^-} \cap \overline{B^+}.$$

This is a contradiction. Similarly, if the case (b) or case (c) occurs, we also get the contradiction.

By the assertion (i), $\overline{S_-} \subset S_- \cup S_0$. Since for any $(u_0, v_0) \in S_0$, we have $(u_0, v_0) \notin \overline{S_-}$, which implies that $\overline{S_-} = S_-$. i.e. S_- is closed.

(iii) By assertions (i) and (ii), we have

$$\overline{S_-} \cap \overline{S_+} = S_- \cap \overline{S_+} \subseteq S_- \cap (S_+ \cup S_0) = (S_- \cap S_+) \cup (S_- \cap S_0) = \emptyset,$$

and so S_- and S_+ are separated. □

When S_- and S_+ are separated and

$$S_0 = \{(u, v) \in Y : u = 0,\ \text{or } v = 0,\ \text{or}, u = v = 0\},$$

any non-zero minimizer of $J(u, v)$ on S_- (or S_+) is also a local minimizer of $J(u, v)$ on S, and so will be a critical point of $J(u, v)$ on S and a solution of (1.2) (see Theorem 2.1).

Theorem 4.2. *Suppose that $G(\phi, \psi) < 0$. Then:*

(i) *every minimizer sequence for $J(u, v)$ on S_- is bounded;*
(ii) $\inf_{(u,v)\in S_-} J(u, v) > 0$*;*
(iii) *there exists a minimizer of $J(u, v)$ on S_-.*

Proof. (i) Suppose that $\{(u_n, v_n)\} \in S_-$ is a minimizing sequence. Then there exist $c_1, c_2 \geq 0$ such that

$$L(u_n) = (\alpha + 1)G(u_n, v_n) \to c_1, \tag{4.5}$$
$$R(v_n) = (\beta + 1)G(u_n, v_n) \to c_2. \tag{4.6}$$

Assume that $\{(u_n, v_n)\}$ is unbounded. Without loss of generality, we may assume that u_n is unbounded in $W_0^{1,p}(\Omega)$ and v_n is bounded in $W_0^{1,q}(\Omega)$. Let $\bar{u}_n = \frac{u_n}{\|u_n\|_{1,p}}$,. Dividing (4.5) by $\|u_n\|_{1,p}^p$, we obtain

$$L(\bar{u}_n) = (\alpha + 1)\|u_n\|_{1,p}^{\alpha+1-p} G(\bar{u}_n, v_n) \to 0.$$

But by Lemma 4.2, $L(\bar{u}_n) \geq \delta_1 \|\bar{u}_n\|_{1,p}^p = \delta_1$. This is impossible.

(ii) Obviously, $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$ from Lemma 4.1. Since

$$J(u, v) = \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) G(u, v) \geq 0 \quad \text{on } S_-,$$

we have that $\inf_{(u,v)\in S_-} J(u, v) \geq 0$. We now show that $\inf_{(u,v)\in S_-} J(u, v) > 0$. In fact, if $\inf_{(u,v)\in S_-} J(u, v) = 0$. Let $\{(u_n, v_n)\} \subset S_-$ be a minimizing sequence. Then

$$L(u_n) = (\alpha + 1)G(u_n, v_n) \to 0, \qquad R(v_n) = (\beta + 1)(\alpha + 1)G(u_n, v_n) \to 0.$$

It follows from the conclusion (i) that $\{(u_n, v_n)\}$ is bounded. We may assume that (3.2) holds.

We claim that $(u_0, v_0) \notin S_0$.

In fact, if $(u_0, v_0) = (0, 0)$, i.e. $(u_n, v_n) \to (u_0, v_0)$ in Y, then $(0, 0) \in S_-$ since S_- is closed, which is impossible.

If $u_0 \neq 0$, $v_0 = 0$. By the lower semi-continuity,

$$L(u_0) \leq \lim_{n\to\infty} L(u_n) = 0. \tag{4.7}$$

But like in the proof of Lemma 4.1, we can obtain that there exist $\delta, \sigma > 0$ such that $L(u_0) \geq 0$, $R(v_0) \geq 0$ when $\lambda_1 < \lambda < \lambda_1 + \delta, \mu_1 < \mu < \mu_1 + \sigma$ and $(u_n, v_n) \in S_-$, $(u_n, v_n) \rightharpoonup (u_0, v_0)$. Therefore, $\|u_0\|_{1,p} = \|u_n\|_{1,p}$ and $(u_n, v_n) \to (u_0, 0) \in S_-$. This is a contradiction.

Similarly, we can get the contradiction when $u_0 = 0$, $v_0 \neq 0$.

The claim is true, i.e. $u_0 \neq 0$, $v_0 \neq 0$.

We have $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}) \in B^0$, since $G(u_0, v_0) = 0$ from (4.7). Therefore, $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}) \in \overline{B^+}$. By the lower semi-continuity,

$$L(u_0) \leq \lim_{n\to\infty} L(u_n) = 0, \qquad R(v_0) \leq \lim_{n\to\infty} R(v_n) = 0,$$

which implies $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}) \in \overline{\Lambda^-}$, so

$$\left(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}\right) \in \overline{\Lambda^-} \cap \overline{B^+}.$$

This is impossible. Therefore, $\inf_{(u,v)\in S_-} J(u, v) > 0$.

(iii) Let $\{(u_n, v_n)\}$ be a minimizer sequence on S_-. Then

$$J(u_n, v_n) = \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) G(u_n, v_n) \longrightarrow \inf_{(u,v)\in S_-} J(u, v) > 0.$$

Therefore,

$$L(u_n) = (\alpha+1)G(u_n, v_n) \longrightarrow (\alpha+1)\frac{\inf_{(u,v)\in S_-} J(u, v)}{\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1} > 0, \tag{4.8}$$

$$R(v_n) = (\beta+1)G(u_n, v_n) \longrightarrow (\beta+1)\frac{\inf_{(u,v)\in S_-} J(u, v)}{\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1} > 0. \tag{4.9}$$

By the conclusion (i) we know that $\{(u_n, v_n)\}$ is bounded. Without loss of generality we may assume that (3.2) holds. Then, by (4.8) and (4.9),

$$G(u_0, v_0) = \lim_{n\to\infty} G(u_n, v_n) > 0.$$

Therefore, $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}) \in B^+$. Like in the proof of (ii), we can obtain $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}) \in \overline{\Lambda^+} \cap \overline{B^+}$. This shows that $(t(u_0, v_0)u_0, s(u_0, v_0)v_0) \in S_-$, where t, s are given in (2.7).

We now show that the relation (3.3) holds. Otherwise, three cases occur by the lower semi-continuity:

(a) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$;
(b) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}$;
(c) $\|u_0\|_{1,p} = \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$.

If the case (a) occurs, then

$$L(u_0) < \lim_{n\to\infty} L(u_n) = (\alpha+1) \lim_{n\to\infty} G(u_n, v_n) = (\alpha+1)G(u_0, v_0),$$
$$R(v_0) < \lim_{n\to\infty} R(v_n) = (\beta+1) \lim_{n\to\infty} G(u_n, v_n) = (\beta+1)G(u_0, v_0).$$

So $(t(u_0, v_0), s(u_0, v_0)) \neq (1, 1)$. Since

$$(t(u_0, v_0)u_n, s(u_0, v_0)v_n) \rightharpoonup (t(u_0, v_0)u_0, s(u_0, v_0)v_0),$$

and the map $(t, s) \longrightarrow J(tu_n, sv_n)$ attains its maximum value at $t = s = 1$, we have that

$$J(t(u_0, v_0)u_0, s(u_0, v_0)v_0) < \lim_{n\to\infty} J(t(u_0, v_0)u_n, s(u_0, v_0)v_n) \leq \lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in S_-} J(u, v).$$

This is a contradiction.

Similarly, if the case (b) or case (c) occurs, we also get the contradiction.

It follows from (3.3) that

$$L(u_0) = (\alpha+1)G(u_0, v_0), \qquad R(v_0) = (\beta+1)G(u_0, v_0).$$

Therefore $(u_0, v_0) \in S$. Since (4.8) and (4.9) hold, we know that $G(u_0, v_0) > 0$, which implies $(u_0, v_0) \in S_-$. Also

$$J(u_0, v_0) = \lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in S_-} J(u, v).$$

This shows that (u_0, v_0) is a minimizer of $J(u, v)$ on S_-. $\square$

Theorem 4.3. *Suppose that Λ^- is non-empty but $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$. Then:*

(i) *$J(u, v)$ is bounded below on S^+ and* $\inf_{(u,v)\in S^+} J(u, v) < 0$*;*
(ii) *there exists a minimizer of $J(u, v)$ on S_+.*

Proof. (i) Since $\overline{\Lambda^-} \cap \overline{B^+} = \emptyset$, we have that $\Lambda^- \cap B^-$ and so S_+ must be non-empty. Now, we prove that $J(u, v)$ is bounded below on S_+. For any $\{(u_n, v_n)\} \in S_+$ we have two cases:

Case (1) u_n is unbounded in $W_0^{1,p}(\Omega)$ and $\lim_{n\to\infty} v_n = 0$ or $\lim_{n\to\infty} u_n = 0$ and v_n is bounded in $W_0^{1,q}(\Omega)$.

Without loss of generality, we may assume that u_n is unbounded in $W_0^{1,p}(\Omega)$ and $\lim_{n\to\infty} v_n = 0$. Then there exists $M_1 > 0$ such that $\|v_n\|_{1,q} < M_1$. By the Poincaré inequality, for $C > 0$,

$$J(u_n, v_n) = \left(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\right) R(v_n) \ge -\left(\frac{1}{q} + \frac{\alpha+1}{p(\beta+1)} - \frac{1}{\beta+1}\right) C M_1^q.$$

We also obtain that $J(u_n, v_n) < 0$ and $J(u_n, v_n) \to 0$ as $n \to \infty$.

Case (2) (u_n, v_n) does not satisfy the condition of case (1).

We claim that $\{(u_n, v_n)\}$ is bounded in this case.

Otherwise, $\|(u_n, v_n)\| \to \infty$. There will be three cases that occur:

(a) u_n is not bounded in $W_0^{1,p}(\Omega)$ and v_n is bounded in $W_0^{1,q}(\Omega)$;
(b) u_n is bounded in $W_0^{1,p}(\Omega)$ and v_n is not bounded in $W_0^{1,q}(\Omega)$;
(c) u_n is not bounded in $W_0^{1,p}(\Omega)$ and v_n is not bounded in $W_0^{1,q}(\Omega)$.

Suppose the case (a) occurs. Let $\bar{u}_n = \frac{u_n}{\|u_n\|_{1,p}}$. Dividing $L(u_n) = (\alpha+1)G(u_n, v_n)$ by $\|u_n\|_{1,p}^p$ gives

$$L(\bar{u}_n) = (\alpha+1)\|u_n\|_{1,p}^{\alpha+1-p} \int_\Omega c(x)|\bar{u}_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x < 0. \tag{4.10}$$

As the left hand side is uniformly bounded but the term $\|u_n\|_{1,p}^{\alpha+1-p} \to \infty$, it follows that

$$\lim_{n\to\infty} \int_\Omega c(x)|\bar{u}_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x = 0. \tag{4.11}$$

We now show that if

$$\|u_0\|_{1,p} = \lim_{n\to\infty} \|\bar{u}_n\|_{1,p} = 1, \qquad \|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}, \tag{4.12}$$

it follows from (4.11) that $(u_0, \frac{v_0}{\|v_0\|_{1,q}}) \in B^0$, which implies $(u_0, \frac{v_0}{\|v_0\|_{1,q}}) \in \overline{B^+}$. Since

$$L(u_0) = \lim_{n\to\infty} L(\bar{u}_n) \le 0, \qquad R(v_0) = \lim_{n\to\infty} R(v_n) \le 0,$$

we see that $(u_0, \frac{v_0}{\|v_0\|_{1,q}}) \in \overline{\Lambda^-} \cap \overline{B^+}$. This is impossible.

Suppose (4.12) is not true. Then three sub-cases occur by the lower semi-continuity:

(a_1) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|\bar{u}_n\|_{1,p}$ and $\|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}$;
(a_2) $\|u_0\|_{1,p} = \lim_{n\to\infty} \|\bar{u}_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$;
(a_3) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|\bar{u}_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$.

If the sub-case (a_1) occurs, by (4.10) and (4.11),

$$L(u_0) < \lim_{n\to\infty} L(\bar{u}_n) \le 0.$$

Hence,

$$\left(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}\right) \in \overline{\Lambda^-} \cap \overline{B^+},$$

which is impossible. If the sub-case (a_2) or sub-case (a_3) occurs, we also get a contradiction. If the case (b) or case (c) occurs, we also get a contradiction.

The claim is true.

Since $\{(u_n, v_n)\}$ is bounded, there exists $M_2 > 0$ such that $\|(u_n, v_n)\| < M_2$. Hence, using (3.1),

$$\begin{aligned} J(u_n, v_n) &= \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) G(u_n, v_n) \\ &\geq C_0 \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) \int_\Omega |u_n|^{\alpha+1} |v_n|^{\beta+1} \mathrm{d}x \\ &\geq \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) C_0 C_1 \|u_n\|_{1,p}^{\alpha+1} \|v_n\|_{1,q}^{\beta+1} \\ &\geq \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) C_0 C_1 M_2^{\frac{\alpha+1}{p}} M_2^{\frac{\beta+1}{q}}, \end{aligned}$$

where $C_0 = \inf_{c \in \bar{\Omega}} c(x) < 0$.

We obtain that $J(u_n, v_n)$ is bounded below from case (1) and case (2), which implies $J(u, v)$ is bounded below on S_+ and $\inf_{(u,v)\in S^+} J(u, v)$ exits. Obviously, $\inf_{(u,v)\in S^+} J(u, v) < 0$, since $(u, v) \in S_+$.

(ii) Suppose that $\{(u_n, v_n)\}$ is a minimizer sequence on S_+. Then, as $n \to \infty$,

$$J(u_n, v_n) = \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1\right) G(u_n, v_n) \longrightarrow \inf_{(u,v)\in S_+} J(u, v) < 0.$$

We can obtain that (u_n, v_n) satisfies the condition of case (2). In fact, if (u_n, v_n) satisfies the condition of case (1), then as $n \to \infty$,

$$J(u_n, v_n) = \left(\frac{\alpha+1}{p(\beta+1)} + \frac{1}{q} - \frac{1}{\beta+1}\right) L(v_n) \longrightarrow 0.$$

This is impossible since $\inf_{(u,v)\in S^+} J(u, v) < 0$.

From the claim of case (2) in (i), we know that (u_n, v_n) is bounded and we may assume that (3.2) holds. Therefore

$$G(u_0, v_0) = \lim_{n\to\infty} G(u_n, v_n) < 0, \qquad L(u_0) \leq \lim_{n\to\infty} L(u_n) < 0, \qquad R(v_0) \leq \lim_{n\to\infty} R(v_n) < 0.$$

Hence, $(\frac{u_0}{\|u_0\|_{1,p}}, \frac{v_0}{\|v_0\|_{1,q}}) \in \Lambda^- \cap B^-$, and so $(t(u_0, v_0)u_0, s(u_0, v_0)v_0) \in S_+$, where t, s are given in (2.7).

We claim that (3.3) is true. Otherwise, then three cases occur by the lower semi-continuity:

(a) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$;
(b) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}$;
(c) $\|u_0\|_{1,p} = \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$.

If the case (a) occurs, then

$$\begin{aligned} L(u_0) &< \lim_{n\to\infty} L(u_n) = (\alpha+1) \lim_{n\to\infty} G(u_n, v_n) = (\alpha+1) G(u_0, v_0), \\ R(v_0) &< \lim_{n\to\infty} R(v_n) = (\beta+1) \lim_{n\to\infty} G(u_n, v_n) = (\beta+1) G(u_0, v_0). \end{aligned}$$

It follows that $(t(u_0, v_0), s(u_0, v_0)) \neq (1, 1)$. But this leads to a contradiction because

$$J(t(u_0, v_0)u_0, s(u_0, v_0)v_0) < J(u_0, v_0) \leq \lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in S_+} J(u, v).$$

Similarly, if the case (b) or case (c) occurs, we can also get the contradiction.

In view of (3.3) we have that

$$L(u_0) = (\alpha+1) G(u_0, v_0) < 0, \qquad R(v_0) = (\beta+1) G(u_0, v_0) < 0.$$

Thus, $(u_0, v_0) \in S_+$ and

$$J(u_0, v_0) = \lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in S_+} J(u, v).$$

This result shows that (u_0, v_0) is a minimizer for $J(u, v)$ on S_+. $\square$

Theorem 4.4. *Suppose that $G(\phi, \psi) < 0$. Then there exist $\delta > 0$ and $\sigma > 0$ such that (1.2) has at least two positive solutions whenever $\lambda_1(a) < \lambda < \lambda_1(a) + \delta$, $\mu_1(b) < \mu < \mu_1(b) + \sigma$.*

Proof. When $\lambda > \lambda_1(a)$, $\mu > \mu_1(b)$, we easily get $(\phi, \psi) \in \Lambda^- \cap B^-$. By Lemma 4.1, Theorems 4.2 and 4.3, we know that there exist $\delta > 0$ and $\sigma > 0$ such that when $\lambda_1(a) < \lambda < \lambda_1(a) + \delta$, $\mu_1(b) < \mu < \mu_1(b) + \sigma$, $J(u, v)$ has a minimizer in each of S_- and S_+. As $J(|u|, |v|) = J(u, v)$, we may assume that these minimizers of $J(u, v)$ are positive. From Theorem 4.1(iii) we get that S_- and S_+ are separated and

$$S_0 = \{(u, v) \in Y : u = 0,\ \text{or } v = 0,\ \text{or } u = v = 0\}.$$

It follows that the minimizers of $J(u, v)$ are its local minimizers in S which do not lie in S_0, and so are positive solutions of (1.2) by Theorem 2.1. $\square$

In the following, we investigate the nature of S_+ as $\lambda \searrow \lambda_1$ and $\mu \searrow \mu_1$.

Theorem 4.5. *Suppose that $G(\phi, \psi) < 0$ and $(u_n, v_n) \in S_+$ for $\lambda = \lambda_n$, $\mu = \mu_n$. When $\lambda_n \searrow \lambda_1$ and $\mu_n \searrow \mu_1$, we have $\lim_{n\to\infty}(\frac{u_n}{\|u_n\|_{1,p}}, \frac{v_n}{\|v_n\|_{1,q}}) = (\phi, \psi)$ in Y.*

Proof. We claim that $u_n \to 0$ or $v_n \to 0$ or $\lim_{n\to\infty}(u_n, v_n) = (0, 0)$.

Case (1): u_n is unbounded in $W_0^{1,p}(\Omega)$ and $v_n \to 0$ or $u_n \to 0$ and v_n is unbounded in $W_0^{1,q}(\Omega)$. Obviously, the claim is true.

Case (2): (u_n, v_n) does not satisfy the condition of case (1).

$\{(u_n, v_n)\}$ is bounded in this case by the proof of Theorem 4.3. Without loss of generality, we assume that (3.2) holds. Also, for any n,

$$L(u_n) = (\alpha + 1)G(u_n, v_n) < 0, \qquad R(v_n) = (\beta + 1)G(u_n, v_n) < 0.$$

We now show that the relation (3.3) is true. Suppose otherwise; then three cases occur by the lower semi-continuity:

(a) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$;
(b) $\|u_0\|_{1,p} < \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} = \lim_{n\to\infty} \|v_n\|_{1,q}$;
(c) $\|u_0\|_{1,p} = \lim_{n\to\infty} \|u_n\|_{1,p}$ and $\|v_0\|_{1,q} < \lim_{n\to\infty} \|v_n\|_{1,q}$.

If the case (a) occurs, then

$$\int_\Omega (|\nabla u_0|^p - \lambda_1 a(x)|u_0|^p)\mathrm{d}x < \lim_{n\to\infty} \int_\Omega (|\nabla u_n|^p - \lambda_n a(x)|u_n|^p)\mathrm{d}x \le 0,$$
$$\int_\Omega (|\nabla v_0|^q - \mu_1 b(x)|v_0|^q)\mathrm{d}x < \lim_{n\to\infty} \int_\Omega (|\nabla v_n|^q - \mu_n b(x)|v_n|^q)\mathrm{d}x \le 0.$$

This is a contradiction.

Similarly, if the case (b) or case (c) occurs, we will also get the contradiction.

Using (3.3) we see that

$$\begin{cases} 0 \le \displaystyle\int_\Omega (|\nabla u_0|^p - \lambda_1 a(x)|u_0|^p)\mathrm{d}x = (\alpha + 1)G(u_0, v_0) \le 0, \\ 0 \le \displaystyle\int_\Omega (|\nabla v_0|^q - \mu_1 b(x)|v_0|^q)\mathrm{d}x = (\beta + 1)G(u_0, v_0) \le 0. \end{cases} \tag{4.13}$$

This shows that $u_0 = k_1\phi$, $v_0 = k_2\psi$ for some k_1 and k_2. Note that $G(\phi, \psi) < 0$; it follows also from (4.13) that $k_1 = 0$, or $k_2 = 0$, or $k_1 = k_2 = 0$. Hence, the claim is true.

Suppose (u_n, v_n) satisfies the condition of case (2). Let $\bar{u}_n = \frac{u_n}{\|u_n\|_{1,p}}$, $\bar{v}_n = \frac{v_n}{\|v_n\|_{1,q}}$. We may assume that $(\bar{u}_n, \bar{v}_n) \rightharpoonup (\bar{u}_0, \bar{v}_0)$ and then $G(\bar{u}_n, \bar{v}_n) \to G(\bar{u}_0, \bar{v}_0)$. Clearly,

$$\int_\Omega (|\nabla \bar{u}_n|^p - \lambda_n a(x)|\bar{u}_n|^p)\mathrm{d}x = (\alpha+1)\|u_n\|_{1,p}^{\alpha+1-p}\|v_n\|_{1,q}^{\beta+1}\int_\Omega c(x)|\bar{u}_n|^{\alpha+1}|\bar{v}_n|^{\beta+1}\mathrm{d}x.$$

Since (u_n, v_n) is bounded and $\|u_n\|_{1,p} \to 0$ or $\|v_n\|_{1,q} \to 0$ by the conclusion (i), we see that

$$\lim_{n\to\infty}\int_\Omega (|\nabla \bar{u}_n|^p - \lambda_n a(x)|\bar{u}_n|^p)\mathrm{d}x = 0.$$

We now show that

$$\|\bar{u}_0\|_{1,p} = \lim_{n\to\infty}\|\bar{u}_n\|_{1,p}, \qquad \|\bar{v}_0\|_{1,q} = \lim_{n\to\infty}\|\bar{v}_n\|_{1,q}. \tag{4.14}$$

Suppose otherwise; then three cases occur by the lower semi-continuity:

(a) $\|\bar{u}_0\|_{1,p} < \lim_{n\to\infty}\|\bar{u}_n\|_{1,p}$ and $\|\bar{v}_0\|_{1,q} < \lim_{n\to\infty}\|\bar{v}_n\|_{1,q}$;
(b) $\|\bar{u}_0\|_{1,p} < \lim_{n\to\infty}\|\bar{u}_n\|_{1,p}$ and $\|\bar{v}_0\|_{1,q} = \lim_{n\to\infty}\|\bar{v}_n\|_{1,q}$;
(c) $\|\bar{u}_0\|_{1,p} = \lim_{n\to\infty}\|\bar{u}_n\|_{1,p}$ and $\|\bar{v}_0\|_{1,q} < \lim_{n\to\infty}\|\bar{v}_n\|_{1,q}$.

If the case (a) occurs, then

$$\int_\Omega (|\nabla \bar{u}_0|^p - \lambda_1 a(x)|\bar{u}_0|^p)\mathrm{d}x < \lim_{n\to\infty}\int_\Omega (|\nabla \bar{u}_n|^p - \lambda_n a(x)|\bar{u}_n|^p)\mathrm{d}x = 0,$$
$$\int (|\nabla \bar{v}_0|^q - \mu_1 b(x)|\bar{v}_0|^q)\mathrm{d}x < \lim_{n\to\infty}\int (|\nabla \bar{v}_n|^q - \mu_n b(x)|\bar{v}_n|^q)\mathrm{d}x = 0.$$

This is impossible.

Similarly, if the case (b) or case (c) occurs, we shall also get the contradiction.

Form (4.14) we have that $\|\bar{u}_0\|_{1,p} = \|\bar{v}_0\|_{1,q} = 1$ and

$$\int_\Omega (|\nabla \bar{u}_0|^p - \lambda_1 a(x)|\bar{u}_0|^p)\mathrm{d}x = \int_\Omega (|\nabla \bar{v}_0|^q - \mu_1 b(x)|\bar{v}_0|^q)\mathrm{d}x = 0.$$

Thus, $\bar{u}_0 = \phi$, $\bar{v}_0 = \psi$.

If (u_n, v_n) satisfies the condition of case (1), we make a small variation and obtain $\bar{u}_0 = \phi$, $\bar{v}_0 = \psi$. The proof is complete. □

5. Non-existence of positive solutions

In this section, we investigate the nature of the Nahari manifold under these hypotheses and show why minimizers may give rise to only the zero solution.

First we consider the case $\int_\Omega c(x)\phi^{\alpha+1}\psi^{\beta+1}\mathrm{d}x > 0$. Then $(\phi, \psi) \in \Lambda^- \cap B^+$ when λ and μ are just greater than $\lambda_1(a)$ and $\mu_1(b)$ respectively. It can be proved like in the proof of Lemma 4.1 that there exist $\delta, \sigma > 0$ such that

$$\overline{\Lambda^-} \subset B^+ \quad \text{if } \lambda_1(a) \le \lambda < \lambda_1(a) + \delta, \qquad \mu_1(b) \le \mu < \mu_1(b) + \sigma.$$

Consequently, $\Lambda^- \cap B^- = \emptyset$ and S_+ is empty. On the other hand, the set S_- is non-empty. However, from the following theorem and corollary we see that there is no positive solution in S_-.

Theorem 5.1. *If $\Lambda^- \cap B^+ \neq \emptyset$, then $\inf_{(u,v)\in S_-} J(u, v) = 0$.*

Proof. Let $(u, v) \in \Lambda^- \cap B^+$. So it is possible to choose $(h, k) \in Y$ with sufficiently small sup norm but sufficiently large Y norm that:

(i) $\int_\Omega (|\nabla(u+\epsilon h)|^p - \lambda a|u+\epsilon h|^p)\mathrm{d}x > 0$, $\int_\Omega (|\nabla(v+\delta k)|^q - \mu b|v+\delta k|^q)\mathrm{d}x > 0$,
(ii) $\int_\Omega c(x)|u+\epsilon h|^{\alpha+1}|v+\delta h|^{\beta+1}\mathrm{d}x > \frac{1}{2}G(u, v)$ for $0 < \epsilon, \delta \le 1$.

Let

$$u_\epsilon = \frac{u + \epsilon h}{\|u + \epsilon h\|_{1,p}}, \qquad v_\delta = \frac{v + \delta k}{\|v + \delta k\|_{1,q}}.$$

Then $(u_0, v_0) \in \Lambda^-$, $(u_1, v_1) \in \Lambda^+$ and a simple continuity argument shows that there exists $0 < \epsilon_0 < 1, 0 < \delta_0 < 1$ such that $(u_{\epsilon_0}, v_{\delta_0}) \in \Lambda^0$. Moreover, there exists a sequence $\{(u_n, v_n)\} \subset \Lambda^+ \cap B^+$ (where $u_n = u_{\epsilon_n}$, $v_n = v_{\delta_n}$) such that

$$\int_\Omega (|\nabla u_n|^p - \lambda a |u_n|^p)\mathrm{d}x \to 0, \qquad \int_\Omega (|\nabla v_n|^q - \mu b |v_n|^q)\mathrm{d}x \to 0,$$

$$\begin{aligned}\int_\Omega c|u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x &= \frac{1}{\|u + \epsilon h\|_{1,p}^{\alpha+1}\|v + \delta k\|_{1,q}^{\beta+1}} \int_\Omega c|u + \epsilon h|^{\alpha+1}|v + \delta k|^{\beta+1}\mathrm{d}x \\ &\geq \frac{1}{2(\|u\|_{1,p} + \|h\|_{1,p})^{\alpha+1}(\|v\|_{1,q} + \|k\|_{1,q})^{\beta+1}} \int_\Omega c|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \frac{\left(\int_\Omega (|\nabla u_n|^p - \lambda a|u_n|^p)\mathrm{d}x\right)^{(\alpha+1)q} \left(\int_\Omega (|\nabla v_n|^q - \mu b|v_n|^q)\mathrm{d}x\right)^{(\beta+1)p}}{\left(\int_\Omega c|u_n|^{\alpha+1}|v_n|^{\beta+1}\mathrm{d}x\right)^{pq}} = 0.$$

This shows that $(t(u_n, v_n)u_n,\ s(u_n, v_n)v_n) \in S_-$, and

$$\begin{aligned}J(t(u_n, v_n)u_n, s(u_n, v_n)v_n) &= K \frac{\left(\int_\Omega |\nabla u_n|^p - \lambda a|u_n|^p\right)^{\frac{(\alpha+1)q}{d}} \left(\int_\Omega (|\nabla v_n|^q - \mu b|v_n|^q)\right)^{\frac{(\beta+1)p}{d}}}{\left(\int_\Omega c|u_n|^{\alpha+1}|v_n|^{\beta+1}\right)^{\frac{pq}{d}}} \\ &\longrightarrow 0.\end{aligned}$$

Hence $\inf_{(u,v)\in S_-} J(u, v) = 0$. □

Theorem 5.2. *Suppose that* $\int_\Omega c(x)\phi^{\alpha+1}\psi^{\beta+1}\mathrm{d}x > 0$. *Then* $\inf_{(u,v)\in S_-} J(u, v) = 0$ *for all* $\lambda > \lambda_1(a)$, $\mu > \mu_1(b)$.

Proof. Since

$$\int_\Omega (|\nabla\phi|^p - \lambda a(x)\phi^p)\mathrm{d}x = (\lambda_1(a) - \lambda)\int_\Omega a(x)\phi^p\mathrm{d}x,$$

$$\int_\Omega (|\nabla\psi|^q - \mu b(x)\psi^q)\mathrm{d}x = (\mu_1(a) - \mu)\int_\Omega b(x)\psi^q\mathrm{d}x,$$

we have that $(\phi, \psi) \in \Lambda^-$ if $\lambda > \lambda_1(a)$, $\mu > \mu_1(b)$. So $(\phi, \psi) \in \Lambda^- \cap B^+$. □

References

[1] P. Drabek, S.I. Pohozaev, Positive solutions for the p-Laplacian: Application of the fibering method, Proc. Roy. Soc. Edinburgh A127 (1997) 703–726.

[2] Y. Bozhkov, E. Mitidieri, Existence of multiple solutions for quasilinear systems via fibering method, J. Differential Equations 190 (2003) 239–267.

[3] K.J. Brown, Y.P. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, J. Differential Equations 193 (2003) 481–499.

[4] Z. Nehari, On a class of nonlinear second-order differential equations, Trans. Amer. Math. Soc. 95 (1960) 101–123.

[5] J.G. Azorero, I.P. Alonso, Some results about the existence of a second positive solution in a quasilinear critical problem, Indiana Univ. Math. J. 43 (1994) 941–957.

[6] H. Berestycki, I. Capuzzo Dolcetta, L. Nirenberg, Variational methods for indefinite superlinear homogeneous elliptic problems, Nonlinear Differential Equations Appl. 2 (1995) 553–572.

[7] P.A. Binding, P. Drabek, Y.X. Huang, On Neumann boundary value problems for some quasilinear elliptic equations, Electron. J. Differential Equations 5 (1997) 1–11.

[8] P.A. Binding, P. Drabek, Y.X. Huang, Existence of multiple solutions of critical quasilinear elliptic Neumann problems, Nonlinear Anal. TMA 42 (2000) 613–629.

[9] Z.M. Guo, Some existence and multiplicity results for a class of quasilinear elliptic eigenvalue problems, Nonlinear Anal. TMA 10 (1992) 957–971.

[10] H. Amann, J.L. Gómez, A priori bounds and multiple solutions for superlinear indefinite elliptic problems, J. Differential Equations 146 (1998) 336–374.
[11] K. de Thélin, J. Vélin, Existence and non-existence of nontrivial solutions for some nonlinear elliptic systems, Rev. Mat. Univ. Complutense Madrid 6 (1993) 153–164.
[12] E. Mitidieri, G. Sweers, R. van der Vorst, Non-existence theorems for systems of quasilinear partial differential equations, Differential Integral Equations 8 (6) (1995) 1331–1354.
[13] Ph. Clément, J. Fleckinger, E. Mitidieri, F. de Thélin, Existence of positive solutions for quasilinear elliptic systems, J. Differential Equations 166 (2000) 455–477.
[14] P. Lundqvist, On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. 109 (1990) 157–164.
[15] M. Willem, Minimax Theorems, Birkhauser, Boston, Basel, Berlin, 1996.